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Nonlinear Rayleigh–Taylor Instability in Magnetic Fluids Between Two Parallel Plates

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A nonlinear stage of the two-dimensional Rayleigh–Taylor instability for two magnetic fluids of finite thickness is studied by including the effect of surface tension between the two fluids. The system is subjected to a tangential magnetic field. The method of multiple scale perturbations is used in order to obtain uniformly valid expansions near the cutoff wavenumber separating stable and unstable deformations. Two nonlinear Schrödinger equations are obtained, one of which leads to the determination of the cutoff wavenumber. The other Schrödinger equation is used to analyze the stability of the system. It is found that if a finite-amplitude disturbance is stable, then a small modulation to the wave is also stable. It is also found that the tangential magnetic field plays a dual role in the stability criterion. Finally, the magnetic permeability constants of the fluid affect the stability conditions.

1. INTRODUCTION

The Rayleigh–Taylor instability has been an important subject of research because of its implications in stellar and planetary interiors.

The problem of Rayleigh–Taylor instability deals with a heavy fluid supported by a light fluid (Chandrasekhar, 1961). As gravity destabilizes the interface, this configuration is unstable. But if surface tension exists between the two fluids, it has a stabilizing effect on the configuration. On the other hand, Zelazo and Melcher (1969) pointed out that a magnetic field applied tangentially to the interface between two kinds of ferrofluids exerts a stabilizing influence on the configuration. It is interesting that the magnetic field is used to stabilize the interface of the fluids and to support the heavy fluid. The stability of a magnetic fluid column was experimentally demonstrated by Zelazo and Melcher (1969).

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Many theoretical and numerical studies have been done to understand the nonlinear stage of the Rayleigh–Taylor instability (e.g., Mohamed and El Shehawey, 1983; Huba *et al.*, 1987; Oron and Rosenau, 1989; Malik and Singh, 1989; Saasen and Tyvand, 1990; Hassam and Huba, 1990; and Iizuka and Wadati, 1990). It is, however, very difficult to deal with the motion of the interface without simplifications. There seem to be no conclusive compact equations for the interface dynamics. Especially when the deformation in the vertical direction gets large, the motion of a bubble occurs. Thus, we encounter a difficulty that the interface as a function of the horizontal coordinates will be multivalued. Some numerical calculations of bubble formation have been reported (Huba *et al.*, 1987; Malik and Singh, 1989).

Malik and Singh (1989) studied the motion of an inviscid, incompressible, nonconducting ferrofluid with a magnetic field and the surface tension under gravity, and demonstrated the formation of bubbles by means of Lagrangian transformations. They showed as well how the magnetic field and the surface tension stabilize the interface to conserve the contours. The formation of bubbles can be inhibited by using a magnetic fluid with a higher permeability and (or) by increasing the strength of the applied magnetic field. In their analysis, they obtained the KdV equation, but its validity seems to be doubtful.

Recently Iizuka and Wadati (1990) studied the two-dimensional nonlinear Rayleigh–Taylor instability with the effect of surface tension between the two fluids, but in the absence of a magnetic field. In their analysis, the stabilizing effect comes only from the surface tension between the two fluids. They obtained three types of nonlinear evolution equations for the interface by means of the reductive perturbation method. Each equation is valid within a certain region of the wavenumber k introduced in the linearized theory. When k is sufficiently large and thus in the stable region, they obtained the nonlinear Schrödinger equation. When k is nearly equal to the critical wavenumber k_c, they obtained the unstable nonlinear Schrödinger equation. Finally, they obtained the nonlinear diffusion equation in the unstable region, where k is smaller than k_c.

In this paper, a nonlinear stage of the Rayleigh–Taylor instability in magnetic fluids is discussed from the viewpoint of the nonlinear wave theory. We present a nonlinear evolution equation for the interface in the presence of a uniform tangential magnetic field.

In Section 2, we formulated the problem and outline the procedure for deriving linear as well as a hierarchy of nonlinear partial differential equations of various orders with the use of the method of multiple scales. The equation governing the evolution of the amplitude is derived in the same section. In Section 3, we derive two nonlinear Schrödinger equations valid for a progressive wave train and stationary waves. The nonlinear cutoff wavenumber which separates the regions of instability from those of stability is arrived at in Section 4. The stability conditions of the system are determined in Section 5.

2. STATEMENT OF THE PROBLEM

Consider two inviscid, incompressible, superposed magnetic fluids of densities ρ_1, ρ_2, magnetic permeabilities μ_1, μ_2, and thicknesses h_1, h_2, respectively. Surface tension exists between the two fluids. The two fluids are influenced by a constant magnetic field H_0 in the x direction. The magnetic fluid is assumed to be initially quiescent with linear magnetization properties.

We assume that the system is two-dimensional. In coordinates (x, y), a periodic wave of wavelength λ propagates in the x direction and gravity g acts in the negative y direction. The x axis is the mean level of the wave, and the flow is bounded by horizontal planes at $y=-h_1$ and $y=h_2$. The interface between the two fluids is described as $y=\eta(x, t)$, where t is the time. When it is completely flat, $\eta=0$.

We shall investigate the propagation of weakly nonlinear waves, confining ourselves to a wave train that has a principal direction along the x axis, although modulations in the x direction will be allowed. As the motion of the system starts from rest, it is taken to be an irrotational flow. The basic equations governing the irrotational motion are

$$\nabla^2\phi_1=0 \qquad \text{for} \quad -h_1<y<\eta(x, t) \tag{2.1}$$

$$\nabla^2\phi_2=0 \qquad \text{for} \quad \eta(x, t)<y<h_2 \tag{2.2}$$

where $\phi(x, y, t)$ is the velocity potential ($\mathbf{v}=\nabla\phi$) and $y=\eta(x, t)$ is the elevation of the free surface.

We also assume that the quasistatic approximation is valid and we introduce the magnetic scalar potential $\psi(x, y, t)$ such that

$$\mathbf{H}_j=H_0\mathbf{e}_x-\nabla\psi_j, \qquad j=1,2$$

where $\mathbf{e}_x$ is the unit vector along the x direction.

Therefore the differential equations satisfied by ψ_j are the Laplace equations

$$\nabla^2\psi_1=0 \qquad \text{for} \quad -h_1<y<\eta(x, t) \tag{2.3}$$

$$\nabla^2\psi_2=0 \qquad \text{for} \quad \eta(x, t)<y<h_2 \tag{2.4}$$

The subscripts 1 and 2 refer to quantities in the lower fluid and upper fluid, respectively.

The solutions for ϕ_j and ψ_j $(j=1,2)$ have to satisfy the following boundary conditions.

2.1. Boundary Conditions

(a) On the rigid boundaries

$$\left.\frac{\partial\phi_1}{\partial y}\right|_{y=-h_1}=0 \qquad \left.\frac{\partial\phi_2}{\partial y}\right|_{y=h_2}=0, \qquad \left.\frac{\partial\psi_1}{\partial x}\right|_{y=-h_1}=0, \qquad \left.\frac{\partial\psi_2}{\partial x}\right|_{y=h_2}=0 \quad (2.5)$$

(b) At the interface $y=\eta(x, t)$:

(i) The kinematic condition is

$$\frac{\partial\phi_j}{\partial y}-\frac{\partial\eta}{\partial t}=\frac{\partial\phi_j}{\partial x}\frac{\partial\eta}{\partial x}, \qquad j=1,2 \quad (2.6)$$

(ii) The tangential component of the magnetic field should be continuous at the interface,

$$\left\{\left|\frac{\partial\psi}{\partial x}\right|\right\}+\frac{\partial\eta}{\partial x}\left\{\left|\frac{\partial\psi}{\partial y}\right|\right\}=0 \quad (2.7)$$

where $\{|\cdot|\}$ represents the jump across the interface.

(iii) Since there are no free currents at the surface $y=\eta(x, t)$, the normal component of the magnetic induction is continuous at the interface,

$$\left\{\left|\mu\frac{\partial\psi}{\partial y}\right|\right\}+H_0(\mu_2-\mu_1)\frac{\partial\eta}{\partial x}=\frac{\partial\eta}{\partial x}\left\{\left|\mu\frac{\partial\psi}{\partial x}\right|\right\} \quad (2.8)$$

(iv) The continuity of normal stresses across $y=\eta(x, t)$ requires

$$(\rho_1-\rho_2)g\eta+\rho_1\frac{\partial\phi_1}{\partial t}-\rho_2\frac{\partial\phi_2}{\partial t}+\frac{1}{2}[\rho_1(\nabla\phi_1)^2-\rho_2(\nabla\phi_2)^2]$$

$$-\sigma\frac{\partial^2\eta}{\partial x^2}\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]^{-3/2}=\frac{1}{2}\{|\mu(H_n^2-H_t^2)|\} \quad (2.9)$$

where σ is the coefficient of surface tension and H_n and H_t represent the normal and tangential components of the magnetic field, respectively.

2.2. Method of Solution and Analysis

The set of equations (2.1)-(2.9) will be solved using the method of multiple scales (Nayfeh, 1976). We expand all the physical quantities in powers of a small parameter ε characterizing the steepness ratio of the wave. The independent variables x, t are scaled in a similar way,

$$X_n=\varepsilon^n x, \qquad T_n=\varepsilon^n t, \qquad n=0,1,2 \quad (2.10)$$

and the variables may be expanded as

$$\eta(x; t) = \sum_{n=1}^{3} \varepsilon^n \eta_n(X_0, X_1, X_2; T_0, T_1, T_2) + O(\varepsilon^4) \tag{2.11}$$

$$\phi_j(x; y; t) = \sum_{n=1}^{3} \varepsilon^n \phi_{jn}(X_0, X_1, X_2; y; T_0, T_1, T_2) + O(\varepsilon^4) \tag{2.12}$$

$$\psi_j(x; y; t) = \sum_{n=1}^{3} \varepsilon^n \psi_{jn}(X_0, X_1, X_2; y; T_0, T_1, T_2) + O(\varepsilon^4) \tag{2.13}$$

where η_1 is expressed in the form

$$\begin{aligned}\eta_1 = {} & A(X_1, X_2; T_1, T_2) \exp[i(kX_0 - \omega T_0)] \\ & + \bar{A}(X_1, X_2; T_1, T_2) \exp[-i(kX_0 - \omega T_0)]\end{aligned} \tag{2.14}$$

Here $k = 2\pi/\lambda$ is the wavenumber, ω is the frequency of the disturbance, and A is an unknown slowly varying function of the amplitude of the propagating wave and will be determined later by the solvability conditions; the bar denotes the complex conjugate.

The boundary conditions (2.6)–(2.9) are prescribed at the perturbed surface $y = \eta(x, t)$. We expand the physical quantities involved in Maclaurin series about $y = 0$. On substituting (2.10)–(2.13) into (2.1)–(2.9) and equating the coefficients of equal powers in ε, we obtain the linear as well as successive higher-order equations, each of which can be solved with the knowledge of the solutions of the previous orders. The procedure is straightforward but lengthy and will not be included here. The details are available from the author and is outlined by Nayfeh (1976).

2.3. Derivation of the Amplitude Equation

The solution of the first-order or linear problem leads to the dispersion relation derived by Rosensweig (1985),

$$\begin{aligned}D(\omega, k) = {} & g(\rho_1 - \rho_2) + H_0^2 \delta(k) k + \sigma k^2 \\ & - \frac{\omega^2}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) = 0\end{aligned} \tag{2.15}$$

where

$$\delta(k) = (\mu_1 - \mu_2)^2 / \tilde{\mu}_0(k)$$

with

$$\tilde{\mu}_0(k) = \mu_1 \coth kh_1 + \mu_2 \coth kh_2$$

From the above dispersion relation, we observe that the magnetic field has a stabilizing influence on the wave motion. This theoretical result was first obtained and confirmed experimentally by Zelazo and Melcher (1969). Since our aim is to study amplitude modulation of the progressive waves, we assume $\omega^2 > 0$ and proceed to the higher-order problems. For the second-order problem, we have the condition

$$-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial T_1}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial X_1}=0 \tag{2.16}$$

and its complex conjugate relation. If $\partial D/\partial\omega \neq 0$, the above condition becomes

$$\frac{\partial A}{\partial T_1}+v_g\frac{\partial A}{\partial X_1}=0 \tag{2.17}$$

where

$$\begin{aligned} v_g = d\omega/dk = \{&\omega^2[\rho_1 h_1 \operatorname{cosech}^2 kh_1 + \rho_2 h_2 \operatorname{cosech}^2 kh_2 \\ &+ (\rho_1 \coth kh_1 + \rho_2 \coth kh_2)/k] \\ &+ H_0^2\delta(k)k[1 + k(\mu_1 h_1 \operatorname{cosech}^2 kh_1 \\ &+ \mu_2 h_2 \operatorname{cosech}^2 kh_2)/\tilde{\mu}_0(k)] + 2\sigma k^2\} \\ &\times [2\omega(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)]^{-1} \end{aligned}$$

is the group velocity of linearized wave theory. It follows as usual that the amplitude A depends on the variables X_1, T_1 through the combination $(X_1 - v_g T_1)$.

If we carry the problem to the third-order set of equations, we can substitute the solutions of the first- and second-order problems into that of the third order and solve the resulting equations. The solutions yield the solvability condition

$$\begin{aligned} &i\left(-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial T_2}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial X_2}\right) \\ &\quad+\frac{1}{2}\left(\frac{\partial^2 D}{\partial \omega^2}\frac{\partial^2 A}{\partial T_1^2}-2\frac{\partial^2 D}{\partial \omega\,\partial k}\frac{\partial^2 A}{\partial X_1\,\partial T_1}+\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial X_1^2}\right)=JA^2\bar{A} \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} J = 2k\omega^2&\left[\rho_1 \coth kh_1\left(1-\frac{1}{\sinh^2 kh_1}\right)+\rho_2 \coth kh_2\left(1-\frac{1}{\sinh^2 kh_2}\right)\right] \\ &-2\delta_0(k)H_0^2k^3 - 1.5\sigma k^4 \\ &-2\left\{\omega^2\left[\rho_1\left(\coth^2 kh_1+\frac{1}{2\sinh^2 kh_1}\right)-\rho_2\left(\coth^2 kh_2+\frac{1}{2\sinh^2 kh_2}\right)\right]\right. \\ &\left.+\delta_1(k)H_0^2k^2\right\}^2 \Big/ D(2\omega, 2k) \end{aligned}$$

with

$$\delta_0(k)=2\delta(k)-\frac{2\delta(k)\delta(2k)}{(\mu_1-\mu_2)^2}+\frac{\delta^2(k)}{(\mu_1-\mu_2)\tilde{\mu}(2k)}(\mu_1\coth kh_1\coth 2kh_1$$
$$-\mu_2\coth kh_2\coth 2kh_2)$$
$$+\frac{\mu_1\mu_2\delta(2k)}{\tilde{\mu}_0^2(k)}(\coth kh_1+\coth kh_2)$$
$$\times[\coth 2kh_1+\coth 2kh_2-2\coth 2kh_1\coth 2kh_2$$
$$\times(\coth kh_1+\coth kh_2)]$$

$$\delta_1(k)=2(\mu_1-\mu_2)+\frac{\delta^2(k)}{2(\mu_2-\mu_1)}+\frac{2\delta(k)\delta(2k)}{\mu_2-\mu_1}$$
$$+\frac{\delta(k)}{2\mu_0(k)}(\mu_2\coth^2 kh_2-\mu_1\coth^2 kh_1)$$
$$+\frac{2\mu_1\mu_2\delta(k)}{(\mu_2-\mu_1)\mu_0(2k)}(\coth kh_1+\coth kh_2)$$
$$\times(\coth 2kh_1+\coth 2kh_2)$$

We notice that the asymptotic expansions break down when the denominator of J equals zero, which corresponds to second-harmonic resonance. We therefore assume that $D(2\omega, 2k)\neq 0$.

3. NONLINEAR SCHRÖDINGER EQUATIONS

The solvability conditions (2.16) and (2.18) can be simplified and combined to produce a single equation. This can be done through some manipulation using the original variables x and t. Finally, we get

$$i\left(\frac{\partial A}{\partial x}+\frac{dk}{d\omega}\frac{\partial A}{\partial t}\right)-\frac{1}{2}\frac{d^2k}{d\omega^2}\frac{\partial^2 A}{\partial t^2}=\left(\varepsilon^2 J\bigg/\frac{\partial D}{\partial k}\right)A^2\bar{A} \tag{3.1}$$

where $dk/d\omega$ is the inverse of the group velocity.

If we use the Gardner–Morikawa transformation

$$\eta=\varepsilon^2 x, \qquad \tau=\varepsilon\left(t-\frac{dk}{d\omega}x\right) \tag{3.2}$$

then equation (3.1) takes the form

$$i\frac{\partial A}{\partial \eta}+P_0\frac{\partial^2 A}{\partial \tau^2}=Q_0A^2\bar{A} \tag{3.3}$$

where

$$P_0=-\frac{1}{2}\frac{d^2k}{d\omega^2} \qquad \text{and} \qquad Q_0=J\bigg/\frac{\partial D}{\partial k}$$

Equation (3.3) is a nonlinear Schrödinger equation. Its solution is valid near $\omega=0$ and therefore can be used to obtain the cutoff wavenumber.

As suggested by Davey (1972) and Nayfeh (1976), we can obtain an equation analogous to (3.1) which contains a second-order space derivative by inserting $d\omega/dk$ instead of $dk/d\omega$ into (2.18). With a similar procedure we get

$$i\left(\frac{\partial A}{\partial t}+\frac{d\omega}{dk}\frac{\partial A}{\partial x}\right)+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial x^2}=\left(-\varepsilon^2 J\bigg/\frac{\partial D}{\partial \omega}\right)A^2\bar{A} \tag{3.4}$$

which is the analogous equation to (3.1). Equation (3.4) includes the first- and second-order spatial derivatives but involves a first-order time derivative only.

Changing the independent variables from x and t into

$$X=\varepsilon(x-v_g t), \qquad T=\varepsilon^2 t \tag{3.5}$$

then equation (3.4) tends to become a second nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial T}+P\frac{\partial^2 A}{\partial X^2}=QA^2\bar{A} \tag{3.6}$$

where

$$P=\frac{1}{2}\frac{dv_g}{dk}, \qquad Q=-J\bigg/\frac{\partial D}{\partial \omega}$$

We observe that the solution of equation (3.6) is not valid at $\omega=0$ and thus cannot be used to obtain the cutoff wavenumber. However, one can study the stability of wave trains for perturbations having values of k not very close to the cutoff wavenumber. It is known that the solutions of equation (3.6) are bounded if $PQ>0$. Thus, if the condition $PQ>0$ is satisfied, the finite deformation of the interface is stable and finite-amplitude waves can propagate through the interface.

We note that if we are interested in equation (3.6) only, then the analysis can be considerably simplified by excluding the time scale T_1. Using only the time scale T_2 and using a coordinate system that translates the group velocity, one can easily obtain equation (3.6).

In order to find the surface deflection, we examine the time-dependent solution of equation (3.1), which can be expressed as

$$A = \tfrac{1}{2}a \exp(ist + \text{const}) \tag{3.7}$$

where a is a constant and

$$s = \left\{\frac{dk}{d\omega} - \left[\left(\frac{dk}{d\omega}\right)^2 + 0.5\varepsilon^2 a^2 Q_0 \frac{d^2k}{d\omega^2}\right]^{1/2}\right\} \Big/ \frac{d^2k}{d\omega^2} \tag{3.8}$$

Expanding (3.8) for small ε, we get

$$\begin{aligned} s &= -\varepsilon^2 a^2 Q_0/(4dk/d\omega) \\ &= -\varepsilon^2 a^2 kJ/[8\omega(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)] \end{aligned} \tag{3.9}$$

Substituting for J into s, one obtains A and hence η. The result is

$$\eta(x, t) = \varepsilon a \cos\theta + 0.5\varepsilon^2 a^2 \Lambda \cos 2\theta + O(\varepsilon^3) \tag{3.10}$$

where

$$\begin{aligned} \theta &= kx - (\omega - s)t \\ \Lambda &= \{\omega^2[\rho_2(\coth^2 kh_2 + 0.5\,\text{cosech}^2\, kh_2) \\ &\quad - \rho_1(\coth^2 kh_1 + 0.5\,\text{cosech}^2\, kh_1)] \\ &\quad - \delta_1(k) H_0^2 k^2\}/D(2\omega, 2k) \end{aligned}$$

Equation (3.10) shows the surface elevation of the interface. The solution breaks up when $D(2\omega, 2k) = 0$ $(\Lambda \to \infty)$ since the second-order term becomes larger than the first-order term. When $D(2\omega, 2k) = 0$, we see that

$$\begin{aligned} & g(\rho_1 - \rho_2)[(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\ & \quad -2(\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_2)] \\ & \quad + 2kH_0^2[\delta(2k)(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\ & \quad - \delta(k)(\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_2)] \\ & \quad + 2\sigma k^2[2(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\ & \quad - (\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_2)] = 0 \end{aligned} \tag{3.11}$$

which is the second-harmonic resonance. It is clear that the magnetic field has an effect on the resonance wavenumber. If $h_1 \to \infty$ and $h_2 \to \infty$ (this is the case of two semi-infinite fluid layers), equation (3.11) becomes

$$k^2 = g(\rho_1 - \rho_2)/(2\sigma) \tag{3.12}$$

which is the same as obtained by Elhefnawy (1990).

4. NONLINEAR CUTOFF WAVENUMBER

Equation (3.6) cannot be applied near the cutoff wavenumber, since, as $\omega \to 0$, we find $v_g \to \infty$. Thus, we may use only the set of relations leading to equation (3.3). In the limit as $\omega \to 0$ we find

$$P_0 = -(\rho_1 \coth k_c h_1 + \rho_2 \coth k_c h_2)\{2\sigma k_c^2 + k_c H_0^2 \delta(k_c) \times [1 + k_c(\mu_1 h_1 \operatorname{cosech}^2 k_c h_1 + \mu_2 h_2 \operatorname{cosech}^2 k_c h_2)]\}^{-1} \tag{4.1}$$

and

$$Q_0 = -k_c^3[1.5\sigma k_c + 2\delta_0(k_c) H_0^2]\{2\sigma k_c + H_0^2 \delta(k_c) \times [1 + k_c(\mu_1 h_1 \operatorname{cosech}^2 k_c h_1 + \mu_2 h_2 \operatorname{cosech}^2 k_c h_2)]\}^{-1} \tag{4.2}$$

where the linear cutoff wavenumber k_c is given by the transcendental equation

$$g(\rho_1 - \rho_2) + H_0^2 \delta(k_c) k_c + \sigma k_c^2 = 0 \tag{4.3}$$

We now discuss the stability of the wave train solution of constant amplitude. Writing

$$A(\eta, \tau) = A_0 \exp[i(k\eta - \Omega\tau)] \tag{4.4}$$

and substituting in equation (3.3), we get

$$\Omega^2 = -[(k + Q_0|A_0|^2)/P_0] \tag{4.5}$$

Since P_0 and Q_0 are nonpositive, we require $k < -Q_0|A_0|^2$ for Ω to become imaginary. The nonlinear cutoff wavenumber is therefore given by

$$K_n = k_c + \varepsilon^2 |A_0|^2 k_c^3 [1.5\sigma k_c + 2\delta_0(k_c) H_0^2]\{2\sigma k_c + H_0^2 \delta(k_c) \times [1 + k_c(\mu_1 h_1 \operatorname{cosech}^2 k_c h_1 + \mu_2 h_2 \operatorname{cosech}^2 k_c h_2)]\}^{-1} \tag{4.6}$$

The nonlinear correction to the wavenumber given on the right-hand side of equation (4.6) can be either positive or negative, depending upon the signs of P_0 and Q_0, thus resulting in stability or instability. Moreover, the bandwidth of spectrum is $O(\varepsilon^2)$ in the wavenumber space for standing waves, and the magnetic field changes the range of unstable wavenumbers.

5. STABILITY CONDITIONS

The analysis of this section will be based on equation (3.6). Equation (3.6) describes the modulation of a one-dimensional weakly nonlinear dispersive wave in the presence of an externally applied magnetic field. It is well known that the solutions of this equation are stable if and only if

$$PQ > 0 \tag{5.1}$$

where

$$
\begin{aligned}
P = {} & \frac{k^2}{8\omega^3(\rho_1 \cosh kh_1 + \rho_2 \coth kh_2)^2} \\
& \times \Bigg[\frac{\omega^4}{k^4} \{3[(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\
& + k(\rho_1 h_1 \operatorname{cosech}^2 kh_1 + \rho_2 h_2 \operatorname{cosech}^2 kh_2)]^2 - 4(\rho_1 \coth kh_1 \\
& + \rho_2 \coth kh_2)[\rho_1 \coth kh_1 + \rho_2 \coth kh_2 \\
& + k(\rho_1 h_1 \operatorname{cosech}^2 kh_1 \\
& + \rho_2 h_2 \operatorname{cosech}^2 kh_2) + k^2(\rho_1 h_1^2 \operatorname{cosech}^2 kh_1 \coth kh_1 \\
& + \rho_2 h_2^2 \operatorname{cosech}^2 kh_2 \coth kh_2)]\} \\
& + \frac{2\omega^2}{k^2} \Bigg(2k(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\
& \times \Bigg\{ \sigma + \frac{H_0^2 \delta(k)}{\tilde{\mu}_0(k)} \Bigg[\mu_1 h_1 \operatorname{cosech}^2 kh_1 + \mu_2 h_2 \operatorname{cosech}^2 kh_2 \\
& + \frac{k}{\tilde{\mu}_0(k)} (\mu_1 h_1 \operatorname{cosech}^2 kh_1 \\
& + \mu_2 h_2 \operatorname{cosech}^2 kh_2)^2 - k(\mu_1 h_1^2 \operatorname{cosech}^2 kh_1 \coth kh_1 \\
& + \mu_2 h_2^2 \operatorname{cosech}^2 kh_2 \coth kh_2) \Bigg] \Bigg\} \\
& + [\rho_1 \coth kh_1 + \rho_2 \coth kh_2 \\
& + k(\rho_1 h_1 \operatorname{cosech}^2 kh_1 + \rho_2 h_2 \operatorname{cosech}^2 kh_2)] \\
& \times \Bigg\{ 2\sigma k + \frac{H_0^2 \delta(k)}{\tilde{\mu}_0(k)} [\tilde{\mu}_0(k) \\
& + k(\mu_1 h_1 \operatorname{cosech}^2 kh_1 + \mu_2 h_2 \operatorname{cosech}^2 kh_2)] \Bigg\} \Bigg) \\
& - \Bigg\{ 2\sigma k + \frac{H_0^2 \delta(k)}{\tilde{\mu}_0(k)} [\tilde{\mu}_0(k) \\
& + k(\mu_1 h_1 \operatorname{cosech}^2 kh_1 + \mu_2 h_2 \operatorname{cosech}^2 kh_2)] \Bigg\}^2 \Bigg]
\end{aligned} \tag{5.2}
$$

and

$$Q=\frac{k}{2\omega(\rho_1 \coth kh_1+\rho_2 \coth kh_2)}$$

$$\times\Bigg(2k\omega^2[(\rho_1 \coth kh_1+\rho_2 \coth kh_2)$$

$$-(\rho_1 \operatorname{cosech}^2 kh_1 \coth kh_1+\rho_2 \operatorname{cosech}^2 kh_2 \coth kh_2)]$$

$$-2\delta_0(k)H_0^2k^3-1.5\sigma k^4-\frac{2}{D(2\omega, 2k)}$$

$$\times\{\delta_1(k)H_0^2k^2+\omega^2[(\rho_1 \coth^2 kh_1-\rho_2 \coth^2 kh_2)$$

$$+0.5(\rho_1 \operatorname{cosech}^2 kh_1-\rho_2 \operatorname{cosech}^2 kh_2)]\}^2\Bigg) \tag{5.3}$$

with

$$\omega^2=\frac{k}{\rho_1 \coth kh_1+\rho_2 \coth kh_2}[g(\rho_1-\rho_2)+H_0^2\delta(k)k+\sigma k^2] \tag{5.4}$$

Thus, a finite-amplitude wave propagating through the surface is stable when the condition given by (5.1) is satisfied. This condition depends on H_0, k, g, σ, $\rho_{1,2}$, $h_{1,2}$, and $\mu_{1,2}$. The critical values of these parameters required for stability may be obtained from the equality of condition (5.1), namely

$$PQ=0 \tag{5.5}$$

The last condition (5.5) is given by the vanishing of P and Q. The condition $P=0$ can be written in the form

$$p_1H_0^4+p_2H_0^2+p_3=0 \tag{5.6}$$

while $Q=0$ becomes

$$\frac{p_6H_0^4+p_7H_0^2+p_8}{p_4H_0^2-p_5}=0 \tag{5.7}$$

where the p's are evaluated and the details are given in the Appendix.

We also observe that the condition (5.7) splits into

$$p_6H_0^4+p_7H_0^2+p_8=0 \tag{5.8}$$

and

$$p_4H_0^2-p_5=0 \tag{5.9}$$

From equations (5.6), (5.8), and (5.9) and inequality (5.1) we find that the system is stable provided that the magnetic field satisfies any of the following sets of conditions:

$$p_1H_0^4+p_2H_0^2+p_3>0, \quad p_6H_0^4+p_7H_0^2+p_8>0, \text{ and } p_4H_0^2-p_5>0 \quad (5.10)$$

$$p_1H_0^4+p_2H_0^2+p_3>0, \quad p_6H_0^4+p_7H_0^2+p_8<0, \text{ and } p_4H_0^2-p_5<0 \quad (5.11)$$

$$p_1H_0^4+p_2H_0^2+p_3<0, \quad p_6H_0^4+p_7H_0^2+p_8<0, \text{ and } p_4H_0^2-p_5>0 \quad (5.12)$$

$$p_1H_0^4+p_2H_0^2+p_3<0, \quad p_6H_0^4+p_7H_0^2+p_8>0, \text{ and } p_4H_0^2-p_5<0 \quad (5.13)$$

For the limiting case as $h_1\to\infty$ and $h_2\to\infty$, we find $P=0$ gives

$$B-m_0=0 \quad (5.14)$$

while $Q=0$ gives

$$\frac{(B-m_1)(B-m_2)}{g(\rho_1-\rho_2)-2\sigma k^2}=0 \quad (5.15)$$

where

$$B=(\mu_2-\mu_1)^2H_0^2/(\mu_2+\mu_1)$$

$$m_0=[g^2(\rho_1-\rho_2)^2-6\sigma gk^2(\rho_1-\rho_2)-3\sigma^2k^4]/(4\sigma k^3)$$

$$m_1,\, m_2=-\frac{\alpha_2}{2\alpha_1}\pm\left[\left(\frac{\alpha_2}{2\alpha_1}\right)^2-\frac{\alpha_3}{\alpha_1}\right]^{1/2}$$

$$\alpha_1=2k^2(\rho^*-\mu^*)^2$$

$$\alpha_2=4k\rho^*(\rho^*-\mu^*)[g(\rho_1-\rho_2)+\sigma k^2]$$

$$\alpha_3=\sigma^2k^4(2\rho^{*2}-1)+\sigma gk^2(\rho_1-\rho_2)(4\rho^{*2}-7/2)$$
$$+2g^2(\rho_1-\rho_2)^2(\rho^{*2}+1)$$

$$\rho^*=(\rho_1-\rho_2)/(\rho_1+\rho_2)$$

$$\mu^*=(\mu_1-\mu_2)/(\mu_1+\mu_2)$$

In this case, we find that the system is stable in any of the following cases:

(i) $B<m_0$ and either $B<m_1$ or $B>m_2$, provided that $k^2<g(\rho_1-\rho_2)/2\sigma$.

(ii) $B<m_0$, $m_1<B<m_2$, and $k^2<g(\rho_1-\rho_2)/2\sigma$.

(iii) $B<m_0$ only if both m_1 and m_2 are complex and $k^2>g(\rho_1-\rho_2)/2\sigma$.

(iv) $B>m_0$, the sign of the inequality involving k^2 and $g(\rho_1-\rho_2)/2\sigma$ being reversed in cases (i)-(iii).

6. CONCLUDING REMARKS

We have shown that the evolution of the amplitude of the progressive as well as the standing waves in superposed magnetic fluids of finite thickness is governed by two nonlinear Schrödinger equations, based on the use of the method of multiple scales. One of them contains only first derivatives in time, while the second contains first and second derivatives in time. The first equation is used to show that the stability of uniform wave trains depends on the wavelength, the surface tension, the gravity, the magnetic field, and the mgnetic permeabilities, densities, and thicknesses of the two fluids. The results show that the waves can be unstable against modulation, in the presence of the tangential magnetic field, if the product of the group velocity rate and the nonlinear interaction coefficient is negative.

Although valid for a wide range of wavenumbers, the first Schrödinger equation is invalid near the cutoff conditions separating stable from unstable motions. However, the second Schrödinger equation, which contains first as well as second derivatives in time, is valid near the cutoff wavenumbers. This second equation is used to determine the dependence of the cutoff wavenumber on the disturbance amplitude.

APPENDIX

The values of the coefficients $p_1, p_2, \ldots, p_8$ appearing in equations (5.6) and (5.7) are

$$p_1 = s_1 s_4 \delta^2(k)\,\frac{4\tilde{\mu}_0(k) - s_4/k}{\tilde{\mu}_0^2(k)} + \delta^2(k)\,\frac{2s_1 s_2 + k(3s_2^2 - 4s_1 s_3)}{s_1}$$

$$+ 2\,\frac{\delta^2(k)}{\tilde{\mu}_0(k)}\left\{ s_2\tilde{\mu}_0(k) + k s_2 s_4 + 2ks_1\left[\frac{s_4^2}{\tilde{\mu}_0(k)} - s_5\right]\right\}$$

$$p_2 = \frac{2g(\rho_1 - \rho_2)\delta(k)}{k\tilde{\mu}_0(k)}\left\{ s_1\left[3s_4 + 2k\left(\frac{s_4^2}{\tilde{\mu}_0(k)} - s_5\right)\right.\right.$$

$$\left.\left. + \frac{k\tilde{\mu}_0(k)}{s_1^2}(3s_2^2 - 4s_1 s_3)\right] + s_2[3\tilde{\mu}_0(k) + ks_4]\right\}$$

$$+ \frac{2\sigma\delta(k)}{\tilde{\mu}_0(k)}\left\{ 2\tilde{\mu}_0(k)s_1 + k[s_1 s_4 + 5\tilde{\mu}_0(k)s_2]\right.$$

$$\left. + k^2\left[s_2 s_4 + 2s_1\left(\frac{s_4^2}{\tilde{\mu}_0(k)} - s_5\right) + \frac{\tilde{\mu}_0(k)}{s_1}(3s_2^2 - 4s_1 s_3)\right]\right\}$$

$$p_3 = \frac{g^2(\rho_1-\rho_2)^2}{k^3}\left(-s_1+2ks_2+k^2\,\frac{3s_2^2-4s_1s_3}{s_1}\right)$$

$$+\frac{2\sigma g(\rho_1-\rho_2)}{k}\left(3s_1+4ks_2+k^2\,\frac{3s_2^2-4s_1s_3}{s_1}\right)$$

$$+\sigma^2 k\left(3s_1+6ks_2+k^2\,\frac{3s_2^2-4s_1s_3}{s_1}\right)$$

$$p_4 = 2k[s_1\delta(2k)-s_9\delta(k)]$$

$$p_5 = g(\rho_1-\rho_2)(2s_9-s_1)+2\sigma k^2(s_9-2s_1)$$

$$p_6 = p_4 s_{11}-2s_{13}^2$$

$$p_7 = p_4 s_{10}-p_5 s_{11}-4s_{12}s_{13}$$

$$p_8 = -(p_5 s_{10}+2s_{12}^2)$$

where

$$s_1 = \rho_1 \coth kh_1 + \rho_2 \coth kh_2$$

$$s_2 = \rho_1 h_1 \operatorname{cosech}^2 kh_1 + \rho_2 h_2 \operatorname{cosech}^2 kh_2$$

$$s_3 = \rho_1 h_1^2 \operatorname{cosech}^2 kh_1 \coth kh_1 + \rho_2 h_2^2 \operatorname{cosech}^2 kh_2 \coth kh_2$$

$$s_4 = \mu_1 h_1 \operatorname{cosech}^2 kh_1 + \mu_2 h_2 \operatorname{cosech}^2 kh_2$$

$$s_5 = \mu_1 h_1^2 \operatorname{cosech}^2 kh_1 \coth kh_1 + \mu_2 h_2^2 \operatorname{cosech}^2 kh_2 \coth kh_2$$

$$s_6 = \rho_1 \operatorname{cosech}^2 kh_1 \coth kh_1 + \rho_2 \operatorname{cosech}^2 kh_2 \coth kh_2$$

$$s_7 = \rho_1 \coth^2 kh_1 - \rho_2 \coth^2 kh_2$$

$$s_8 = \rho_1 \operatorname{cosech}^2 kh_1 - \rho_2 \operatorname{cosech}^2 kh_2$$

$$s_9 = \rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_2$$

$$s_{10} = 2k^2(1-s_6/s_1)[g(\rho_1-\rho_2)+\sigma k^2]-\tfrac{3}{2}\sigma k^4$$

$$s_{11} = 2k^3[\delta(k)(1-s_6/s_1)-\delta_0(k)]$$

$$s_{12} = k(s_7+0.5s_8)[g(\rho_1-\rho_2)+\sigma k^2]/s_1$$

$$s_{13} = k^2[\delta_0(k)(s_7+0.5s_8)/s_1+\delta_1(k)]$$

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